

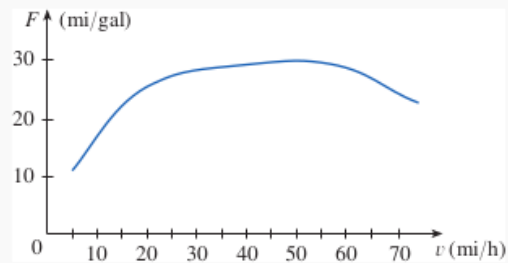
# MATH1001 Lab 4 (Unit 4)

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## Problem 1

The graph (from the US Department of Energy) shows how driving speed affects gas mileage. Fuel economy  $F$  is measured in miles per gallon and speed  $v$  is measured in miles per hour.

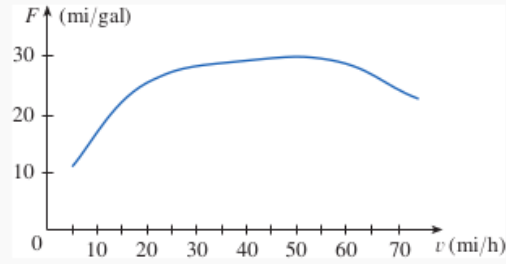
- What is the meaning of the derivative  $F'(v)$ ?
- Sketch the graph of  $F'(v)$ .
- At what speed should you drive if you want to save on gas?



## Solution

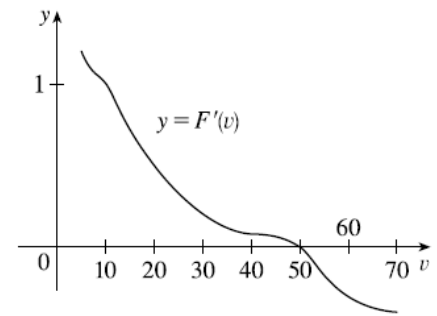
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- What is the meaning of the derivative  $F'(v)$ ?
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## Solution

- $F'(v)$  is the instantaneous rate of change of fuel economy with respect to speed.
- Graphs will vary depending on estimates of  $F'$ , but will change from positive to negative at about  $v = 50$ .
- To save on gas, drive at the speed where  $F$  is a maximum and  $F'$  is 0, which is about 50 mi/h.



## Problem 2

Address the following:

a. Sketch the graph of  $f(x) = 1 + \sqrt{x+3}$

b. Use the graph from [part \(a\)](#) to sketch the graph of  $f'$ .

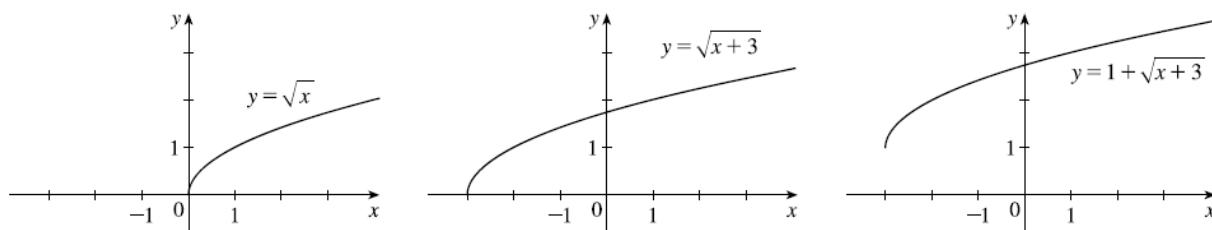
c. Use the definition of a derivative to find  $f'(x)$ . What are the domains of  $f$  and  $f'$ ?

## Solution



## Solution

(a)

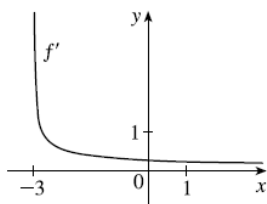


(b) Note that the third graph in part (a) generally has small positive values for its slope,  $f'$ ; but as  $x \rightarrow -3^+$ ,  $f' \rightarrow \infty$ . See the graph in part (d).

$$\begin{aligned}
 \text{(c) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{1 + \sqrt{(x+h)+3} - (1 + \sqrt{x+3})}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)+3} - \sqrt{x+3}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)+3} - \sqrt{x+3}}{h} \left[ \frac{\sqrt{(x+h)+3} + \sqrt{x+3}}{\sqrt{(x+h)+3} + \sqrt{x+3}} \right] \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)+3] - (x+3)}{h(\sqrt{(x+h)+3} + \sqrt{x+3})} = \lim_{h \rightarrow 0} \frac{x+h+3 - x-3}{h(\sqrt{(x+h)+3} + \sqrt{x+3})} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{(x+h)+3} + \sqrt{x+3})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{(x+h)+3} + \sqrt{x+3}} = \frac{1}{2\sqrt{x+3}}
 \end{aligned}$$

Domain of  $f = [-3, \infty)$ , Domain of  $f' = (-3, \infty)$ .

(d)



**Problem 3**

If  $f(x) = x^4 + 2x$ , find  $f'(x)$ .

**Solution**

### Solution

$$\begin{aligned} \text{(a) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[(x+h)^4 + 2(x+h)] - (x^4 + 2x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2x + 2h - x^4 - 2x}{h} \\ &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2h}{h} = \lim_{h \rightarrow 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3 + 2)}{h} \\ &= \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3 + 2) = 4x^3 + 2 \end{aligned}$$

**Problem 4**

If  $f(x) = 2x^2 - x^3$ , find  $f'(x)$ ,  $f''(x)$ ,  $f'''(x)$ , and  $f^{(4)}(x)$ .

**Solution**





## Solution

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[2(x+h)^2 - (x+h)^3] - (2x^2 - x^3)}{h} \\&= \lim_{h \rightarrow 0} \frac{h(4x + 2h - 3x^2 - 3xh - h^2)}{h} = \lim_{h \rightarrow 0} (4x + 2h - 3x^2 - 3xh - h^2) = 4x - 3x^2\end{aligned}$$

$$\begin{aligned}f''(x) &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \rightarrow 0} \frac{[4(x+h) - 3(x+h)^2] - (4x - 3x^2)}{h} = \lim_{h \rightarrow 0} \frac{h(4 - 6x - 3h)}{h} \\&= \lim_{h \rightarrow 0} (4 - 6x - 3h) = 4 - 6x\end{aligned}$$

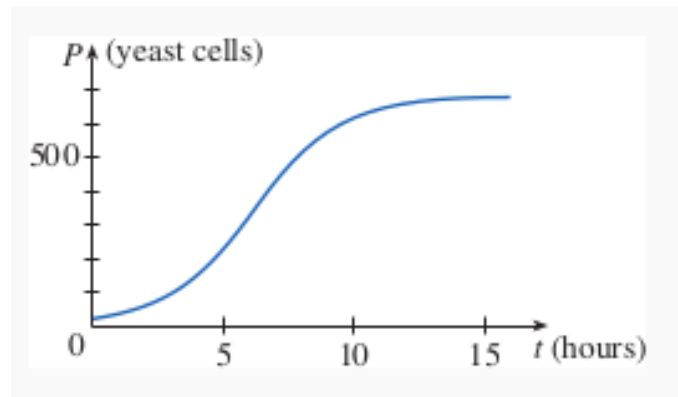
$$f'''(x) = \lim_{h \rightarrow 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h \rightarrow 0} \frac{[4 - 6(x+h)] - (4 - 6x)}{h} = \lim_{h \rightarrow 0} \frac{-6h}{h} = \lim_{h \rightarrow 0} (-6) = -6$$

$$f^{(4)}(x) = \lim_{h \rightarrow 0} \frac{f'''(x+h) - f'''(x)}{h} = \lim_{h \rightarrow 0} \frac{-6 - (-6)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \lim_{h \rightarrow 0} (0) = 0$$

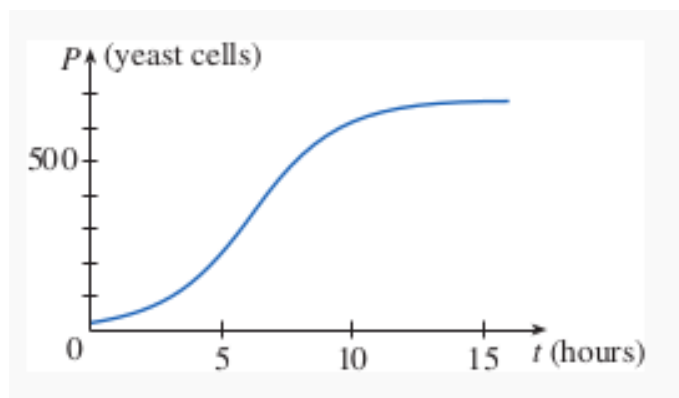
### Problem 5

Shown is the graph of the population function  $P(t)$  for yeast cells in a laboratory culture.

- Sketch the graph of the derivative  $P'(t)$
- Interpret what the graph of  $P'(t)$  tells about the yeast population.

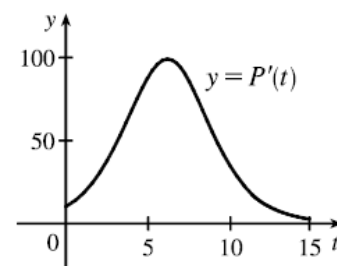


### Solution



## Solution

The slopes of the tangent lines on the graph of  $y = P(t)$  are always positive, so the  $y$ -values of  $y = P'(t)$  are always positive. These values start out relatively small and keep increasing, reaching a maximum at about  $t = 6$ . Then the  $y$ -values of  $y = P'(t)$  decrease and get close to zero. The graph of  $P'$  tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.



**Problem 6**

Find a formula for a function that has vertical asymptotes  $x = 1$  and  $x = 3$ , and horizontal asymptote  $y = 1$ .

**Solution**

## Solution

In general if those are the only requirements, you can take a [rational function](#),

$$f(x) = \frac{P(x)}{Q(x)}$$

where  $P$  and  $Q$  are polynomials. Since you need 2 vertical asymptotes, you can take  $Q(x) = (x-7)(x-9)$  (in general just take a polynomial where the vertical asymptotes are the roots). Now, as for the horizontal asymptote, you can easily prove that if  $\deg(P) = \deg(Q)$  and the leading coefficients for  $P$  and  $Q$  are  $p$  and  $q$  respectively,

$$\lim_{x \rightarrow \infty} \frac{P(x)}{Q(x)} = \frac{p}{q}$$

So in this example,

$$f(x) = \frac{x^2}{(x-1)(x-3)}$$

**Problem 7**

Prove by definition that:

$$\lim_{x \rightarrow 1} \left( \frac{2x^2 - 3x + 1}{x - 1} \right) = 1$$

**Solution**

**Solution**

$$\begin{aligned}\lim_{x \rightarrow 1} \left( \frac{2x^2 - 3x + 1}{x - 1} \right) &= \lim_{x \rightarrow 1} \left( \frac{(2x - 1)(x - 1)}{(x - 1)} \right) \\ &= \lim_{x \rightarrow 1} (2x - 1) \\ &= 1\end{aligned}$$